

An Exact Renormalization Group analysis of $3 - d$ Well Developed turbulence *

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Abstract

We take advantage of peculiar properties of three dimensional incompressible turbulence to introduce a nonstandard Exact Renormalization Group method. A Galilean invariance preserving regularizing procedure is utilized and a field truncation is adopted to test the method. Results are encouraging: the energy spectrum $E(k)$ in the inertial range scales with exponent -1.666 ± 0.001 and the Kolmogorov constant C_K , computed for several (realistic) shapes of the stirring force correlator, agrees with experimental data.

1 Introduction

Exact Renormalization Group (ERG) [1][2][3] represents a powerful tool for a nonperturbative analysis of quantum/statistical systems. It allows us to obtain, via an evolution equation, an effective action Γ_Λ in which only statistical fluctuations of scales smaller than $1/\Lambda$ are computed. Solving this evolution equation for $\Lambda \rightarrow 0$ leads us to the effective action Γ , the

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Legendre transform of the generating functional of connected diagrams [1]. This quite new tool has been applied to QFT [2][4] and critical phenomena, mainly in conjunction with a Derivative Expansion [2], that is a truncation of Γ_Λ up to a maximum number of derivatives.

In this letter we present a modified ERG method for $3 - d$ turbulence and we explore the consequences of a simple field truncation in solving the set of coupled differential nonlinear equations of the ERG flow. The truncation takes into account the effects of the turbulent diffusivity corrections as well as noise corrections due to the energy cascade.

2 The model

Let us consider an isotropic homogeneous model of incompressible turbulence for a Newtonian fluid of viscosity ν and density ρ , where the effects of boundary/initial conditions are taken into account by the introduction of a stochastic noise term $f_\alpha(\vec{x}, t)$. The “stirred” Navier-Stokes equations for the velocity $v_\alpha(\vec{x}, t)$ and pressure $p(\vec{x}, t)$ are

$$\mathcal{D}_t v_\alpha(\vec{x}, t) - \nu \nabla^2 v_\alpha(\vec{x}, t) + \frac{1}{\rho} \partial_\alpha p(\vec{x}, t) = f_\alpha(\vec{x}, t), \quad \partial_\alpha v_\alpha(\vec{x}, t) = 0 \quad (2.1)$$

where the convective derivative $\mathcal{D}_t = (\partial_t + v_\beta \partial_\beta)$ introduces the nonlinear term which generates the turbulent behavior. These equations can be simplified by the introduction of a transverse projector $P_{\alpha\beta}(\nabla) = \left(\delta_{\alpha\beta} - \frac{\partial_\alpha \partial_\beta}{\nabla^2} \right)$ and by the transverse convective derivative $D_t = \partial_t + P(\nabla)(v_\beta \partial_\beta)$:

$$D_t v_\alpha(\vec{x}, t) - \nu \nabla^2 v_\alpha(\vec{x}, t) = f_\alpha(\vec{x}, t). \quad (2.2)$$

The stirring force f is chosen be a Gaussian stochastic white noise field which is completely determined by its two order correlator. Since the stirring force should mimic the instabilities occurring near the boundaries, its correlation length must be of the same order of the size of the system. If L is such a typical length, the correlation function

$$\langle f_\alpha(\vec{q}, \omega) f_\beta(\vec{q}', \omega') \rangle = P_{\alpha\beta}(\vec{q}) F(q) (2\pi)^4 \delta^{(3)}(\vec{q} + \vec{q}') \delta(\omega + \omega') \quad (2.3)$$

must vanish for $q \gg 1/L$ and for $q \ll 1/L$. The precise shape of the spectrum $F(q)$ is not much important, nevertheless a normalization condition due to the energy conservation must be satisfied [5] $\langle \mathcal{E} \rangle = \int \frac{d^3 q}{(2\pi)^3} F(q)$, where \mathcal{E} is the rate of energy dissipated by a unit mass of fluid.

To handle the statistical properties in a easy way and to throw a connection with the usual QFT techniques, we need an action S for such model. Such action is given by the Martin, Siggia and Rose functional [7], written in terms of the transverse field v and its canonical conjugated \hat{u} :

$$\begin{aligned} S[\hat{u}, v] = & \int d^3x dt \hat{u}_\alpha(\vec{x}, t) D_t v_\alpha(\vec{x}, t) - \hat{u}_\alpha(\vec{x}, t) \nu \nabla^2 v_\alpha(\vec{x}, t) \\ & + \frac{i}{2} \int d^3x d^3y dt \hat{u}_\alpha(\vec{x}, t) P_{\alpha\beta}(\nabla) F(x - y) \hat{u}_\beta(\vec{y}, t). \end{aligned} \quad (2.4)$$

The generating functional of connected Green function can be written as

$$\mathcal{W}[\hat{J}, J] = -i \log \left(\int \mathcal{D}\hat{u} \mathcal{D}v \exp(iS[\hat{u}, v] - i\hat{J}v - iJ\hat{u}) \right). \quad (2.5)$$

where $\hat{J}_\alpha(\vec{x}, t)$ and $J_\alpha(\vec{x}, t)$ are two auxiliary sources. We will compute \mathcal{W} as the Legendre transform of the effective action $\Gamma[\hat{u}, v]$, that is

$$\mathcal{W}[\hat{J}, J] = \Gamma[\hat{u}, v] - \hat{J}v - J\hat{u} \quad (2.6)$$

where \hat{u} and v are determined by $\hat{u}_\alpha(\vec{x}, t) = -\frac{\delta \mathcal{W}}{\delta J_\alpha(\vec{x}, t)}$ and $v_\alpha(\vec{x}, t) = -\frac{\delta \mathcal{W}}{\delta \hat{J}_\alpha(\vec{x}, t)}$. The effective action $\Gamma[\hat{u}, v]$ will be obtained by means of the ERG evolution equation.

3 ERG in turbulence

In this section we will recall the main features on ERG method and we will apply it to the simple model of the previous section. The ERG prescription is based on the following points.

- Consider a quantum/statistical theory described in terms of some field ϕ and action $S[\phi]$. For notational simplicity let ϕ be a scalar field. If we modify the system in such a way that no quantum/statistical fluctuations are allowed at distances greater than a certain scale, say, $1/\Lambda_0$ with Λ_0 very high, than the total contribution of the allowed fluctuations is very small (and goes to zero into the limit $\Lambda_0 \rightarrow \infty$). This means that, for this modified dynamics which is described by some new action $S_{\Lambda_0} \equiv S + \Delta S_{\Lambda_0}$, the effective action $\tilde{\Gamma}_{\Lambda_0}$ computed with S_{Λ_0} slightly differs from S_{Λ_0} itself.
- Given ΔS_Λ one can find the evolution equation for the effective action $\tilde{\Gamma}_\Lambda$ with respect to the scale parameter Λ . Let

$$\mathcal{W}_\Lambda[J] = -i \log \left(\int \mathcal{D}\phi \exp(iS_\Lambda[\phi] - iJ\phi) \right) \quad (3.7)$$

be the generating functional of the connected Green function of this modified theory. Following Wetterich [1] we have:

$$\Lambda \partial_\Lambda \tilde{\Gamma}_\Lambda|_\Phi = \Lambda \partial_\Lambda \mathcal{W}_\Lambda|_J = \Lambda \partial_\Lambda < \Delta S_\Lambda[\phi] > \quad (3.8)$$

where $\Phi(\vec{x}, t) = -\frac{\delta \mathcal{W}_\Lambda}{\delta J_\alpha(\vec{x}, t)}$ is the mean field of the modified system. We can introduce also $\Gamma_\Lambda[\Phi]$, the effective action which collapses to S if fluctuations were removed: $\Gamma_\Lambda[\Phi] = \tilde{\Gamma}_\Lambda[\Phi] - \Delta S_\Lambda[\Phi]$. If $\Delta S_\Lambda[\phi]$ is a quadratic term $\Delta S_\Lambda[\phi] = \frac{i}{2} \int d^d x d^d y \phi(x) R_\Lambda(x-y) \phi(y)$, equation (3.8) becomes [1]

$$\Lambda \partial_\Lambda \Gamma_\Lambda|_\Phi = \frac{i}{2} \text{tr} \left(\Lambda \partial_\Lambda R_\Lambda \cdot \left(\frac{1}{\frac{\delta^2 \Gamma_\Lambda}{\delta \Phi \delta \Phi}} + R_\Lambda \right) \right). \quad (3.9)$$

This is the evolution equation we are faced to solve.

- Let us now consider the initial and boundary conditions for Γ_Λ . If we start with a sufficiently high cutoff Λ_0 , fluctuations are strongly suppressed and we can safely say

$$\text{if } \Lambda = \Lambda_0, \text{ then } \Gamma_{\Lambda_0}[\Phi] = S[\Phi].$$

Since our goal is the computation of the “physical” effective action $\Gamma[\Phi]$, that is the sum of all the 1-particle irreducible graphs, we require that

$$\lim_{\Lambda \rightarrow 0} \Gamma_\Lambda[\Phi] = \Gamma[\Phi]. \quad (3.10)$$

Note that when $\lim_{\Lambda \rightarrow 0} \Delta S_\Lambda = 0$ equation (3.10) is satisfied. To sum up, if ΔS_Λ is such that

- 1) for $\Lambda \rightarrow \Lambda_0$ no quantum/statistical fluctuation occurs and
- 2) $\lim_{\Lambda \rightarrow 0} \Delta S_\Lambda = 0$,

then the solution of the evolution equation (3.9) gives us the effective action of the theory.

Let us now apply the method to the isotropic model of turbulence. We firstly must look for ΔS_Λ terms to add to the MSR action in order to satisfy the boundary conditions listed above. To do this, we need to discuss some physical aspects underlying the turbulent system.

The dynamics is characterized by the presence of two typical scales. One of them corresponds to the macroscopic size L of the system. L is the scale at which the energy is introduced into the system. Such energy is transferred to smaller scales via the energy cascade driven by the nonlinear terms [6] [8]. This energy transfer stops at very smaller scales. The second intrinsic scale is the typical scale η_d at which such energy is dissipated by the viscous term. The smaller scale η_d is called the “internal” or the “Kolmogorov” scale. By dimensional arguments it is easy to show (see for example [6]) that the internal scale is related to the physical parameters ν and \mathcal{E} by

$$\eta_d \approx \left(\frac{\nu^3}{\mathcal{E}} \right)^{1/4} \approx \mathcal{R}^{-3/4} L$$

where \mathcal{R} is the Reynolds number of the model $\mathcal{R} = UL/\nu$, being U the typical velocity at the scale L . Since \mathcal{R} is the typical ratio between the inertial term $(U\nabla)U$ and viscous one $\nu\nabla^2U$, it measures the “strength” of the nonlinearity occurring at the scale L .

Turbulent regime can be realized only if \mathcal{R} is much greater than unity. If the Reynolds number is very high, the system develops a wide domain of the Fourier space, called “inertial range”, in which nonlinear terms dominate. Clearly, for momenta $(1/L) \ll q \ll (1/\eta_d) \approx \mathcal{R}^{3/4}(1/L)$ both boundary terms (the stirring force in our model) and viscous terms can be neglected. In this subrange of Fourier space we expect the system to be universal, self similar, homogeneous in space and isotropic [6].

Statistical properties of the inertial range have been studied intensively with many methods (for a review see, for example, [6]). A way to obtain important information on the system is to apply a dimensional analysis (Kolmogorov, 1941 [8]). A simple analysis shows, for example, that the energy spectrum $E(k) = \frac{1}{2} \frac{4\pi}{(2\pi)^3} \text{tr} < u_\alpha(\vec{k}, t) u_\beta(-\vec{k}, t) >$ must follow a power law of the form $E(k) = C_K \mathcal{E}^{2/3} k^{-\frac{5}{3}}$ (Kolmogorov law), where C_K is a constant (Kolmogorov constant) of the order of unity. Clearly, being such analysis essentially a mean field approximation, we expect the exponents of the various moments be corrected by some anomalous dimensions. Such corrections are linked to the intermittent behavior of turbulent systems, that is strong nonlinear, rare, events usually associated to instantons [6].

Let us try to use such informations to construct a regularizing term ΔS_Λ . First of all note that if a system with a Kolmogorov momentum scale $K_d = 1/\eta_d$ is stirred at momentum scales higher than K_d , no energy cascade develops. This can simply be inferred by looking at the Reynolds number $\mathcal{R} \approx (L/\eta_d)^{4/3} \ll 1$. Such a system is then quite non fluctuating and the effective action doesn’t differ too much from the bare action. This gives us the

right initial condition mentioned in point 1). Therefore we can introduce a ΔS_Λ term which modify the characteristic size L of the system into $\eta = 1/\Lambda$, smaller than the internal scale η_d . Since the size L characterizes the force-force correlation function only, the ΔS_Λ term can be chosen as a functional that replaces the stirring force centered at the scale L with a stirring force at the scale $\eta = (1/\Lambda) < \eta_d \ll L$. Given $F(q)$, a possible choice of ΔS_Λ is then

$$\begin{aligned}\Delta S_\Lambda[\phi] &\equiv \frac{i}{2} \int d^3x d^3y dt \hat{u}_\alpha(\vec{x}, t) R_{\Lambda, \alpha\beta} \hat{u}_\beta(\vec{y}, t) \\ &= \frac{i}{2} \int d^3x d^3y dt \hat{u}_\alpha(\vec{x}, t) P_{\alpha\beta}(\nabla) (F_\Lambda(x - y) - F(x - y)) \hat{u}_\beta(\vec{y}, t),\end{aligned}\quad (3.11)$$

where $F_\Lambda(q) \equiv 2D_0\Lambda^{-3}h(q/\Lambda)$ has been obtained from $F(q)$ by a rescaling $L \rightarrow \eta = 1/\Lambda$. The parameter D_0 is linked, via the energy conservation, to the rate of energy dissipation \mathcal{E} . By taking into account equation the normalization equation we have, in fact

$$\int dqq^2h(q/\Lambda) \frac{D_0}{\Lambda^3} = \frac{1}{\pi^2} \mathcal{E}. \quad (3.12)$$

The regularized MSR action of this system $S_\Lambda = S + \Delta S_\Lambda$ is now

$$\begin{aligned}S_\Lambda[\hat{u}, v] &= \int d^3x dt \hat{u}_\alpha(\vec{x}, t) D_t v_\alpha(\vec{x}, t) - \hat{u}_\alpha(\vec{x}, t) \nu \nabla^2 v_\alpha(\vec{x}, t) \\ &\quad + \frac{i}{2} \int d^3x d^3y dt \hat{u}_\alpha(\vec{x}, t) P_{\alpha\beta}(\nabla) F_\Lambda(x - y) \hat{u}_\beta(\vec{y}, t),\end{aligned}\quad (3.13)$$

and the evolution equation (3.9) reads

$$\Lambda \partial_\Lambda \Gamma_\Lambda = \frac{i}{2} \text{tr} \left(\Lambda \partial_\Lambda R_\Lambda \cdot \left(\frac{1}{\tilde{\Gamma}^{(2)}} \right) |_{\hat{u}, \hat{u}} \right). \quad (3.14)$$

In our notation $\left(\frac{1}{\tilde{\Gamma}^{(2)}} \right) |_{\hat{u}, \hat{u}}$ means the full, fields dependent, regularized propagator of the \hat{u} field.

Let us now check the boundary condition mentioned in point 2). The regularizing term (3.11) satisfies $\lim_{\Lambda \rightarrow 0} \Delta S_\Lambda = 0$ in the following sense. Firstly we take the limit $\Lambda \rightarrow 1/L$ so that $\Delta S_{1/L} = 0$ identically. Secondly the infinite volume limit $L \rightarrow \infty$ is considered.

Since our choice of ΔS_Λ satisfies both initial and boundary conditions, the effective action computed by the ERG flow will be, in the limit $\Lambda \rightarrow 1/L \rightarrow 0$, the “physical” effective action of the model.

It should be noted that our choice of the regularizing term ΔS_Λ is somewhat unusual. In all the previous applications of ERG the ΔS_Λ term is introduced in such a way that *all*

the propagators of the theory are damped for $q < \Lambda$. This is not the case in our approach, since the correlator of the regularized system $\langle \hat{u}v \rangle_\Lambda$ has support on the entire Fourier space. Note, however, that all the correlation functions of the *physical fields*, *i.e.* $\langle vv \rangle_\Lambda$ are suppressed for $q < \Lambda$ both at the bare and at the “dressed” level. Indeed, the correlator $\langle vv \rangle_\Lambda \approx \frac{\tilde{\Gamma}_{\hat{u}\hat{u}}}{|\tilde{\Gamma}_{\hat{u}\hat{v}}|^2}$ is proportional to the noise correlation function $\tilde{\Gamma}_{\hat{u}\hat{u}}(p) \approx (F_\Lambda(p) + G_\Lambda(p))$, which contains the correction of the noise term G_Λ . The key point here is that the noise $G_\Lambda(p)$ correction vanishes for $p \ll L$, so that the “dressed” noise term is damped for $q < \Lambda$. This is a consequence of the structure of the nonlinear terms in 3 dimensions and is deeply linked to the features of the energy cascade [17], where the energy is mainly transported from low to high momenta and not *vice-versa* (2-d turbulence and situations which give rise to backscattering [9] are here not taken into account).

4 The ERG flow

The evolution equation (3.14) brings to an infinite system of coupled nonlinear differential equations, one equation for every vertex $\Gamma_\Lambda^{(n)} \equiv \frac{\delta^n \Gamma_\Lambda}{\delta \phi^n}|_{\phi=0}$. Because of the symmetries and of the structure of the ERG flow, however, some restrictions are found. Clearly the MSR action of the systems is invariant under Galilean transformations of an infinitesimal “boost” c_α . Since the regularizing action ΔS_Λ is Galilean invariant too [17], one finds Ward-Takahashi Identities (WT), which must be satisfied by Γ_Λ :

$$\int d^3x dt \left\{ (tc_\beta \partial_\beta v_\alpha(\vec{x}, t) - c_\alpha) \frac{\delta}{\delta v_\alpha(\vec{x}, t)} + tc_\beta \partial_\beta \hat{u}_\alpha(\vec{x}, t) \frac{\delta}{\delta \hat{u}_\alpha(\vec{x}, t)} \right\} \Gamma_\Lambda[\hat{u}, v] = 0. \quad (4.15)$$

For a more detailed analysis of WT identities see [17]. The structure of the evolution equation poses some important restrictions on the form of $\Gamma_\Lambda[\hat{u}, v]$. Indeed, the presence of the $\Lambda \partial_\Lambda R_\Lambda$ term in the evolution equation (3.14) implies that no vertex correction with v legs only can be generated by the ERG flow [17].

Even if some restrictions on the form of $\Gamma_\Lambda[\hat{u}, v]$ are found, the system of ERG differential equations is still of infinite order. In order to handle the ERG equations, in this work we will explore the consequences of a field truncation in Γ_Λ . The simplest truncation of Γ_Λ which preserves WT identities is the following

$$\begin{aligned} \Gamma_\Lambda^{trunc.}[\hat{u}, v] = & \int d^3x d^3y dt \left\{ \delta^{(3)}(x - y) \hat{u}_\alpha(\vec{x}, t) D_t v_\alpha(\vec{y}, t) - \hat{u}_\alpha(\vec{x}, t) f(\Lambda, x - y) \nabla^2 v_\alpha(\vec{y}, t) \right\} \\ & + \frac{i}{2} \int d^3x d^3y dt \hat{u}_\alpha(\vec{x}, t) P_{\alpha\beta}(\nabla) (F_\Lambda(x - y) + g(\Lambda, x - y)) \hat{u}_\beta(\vec{y}, t). \end{aligned} \quad (4.16)$$

This ansatz for Γ_Λ takes into account the noise correction $g(\Lambda, x - y)$ (energy cascade) as well as the damping correction $f(\Lambda, x - y)$. The evolution equation for $\Gamma_\Lambda^{trunc.}$ reduces then to a system of two coupled nonlinear differential equations in terms of $f(\Lambda, x - y)$ and $g(\Lambda, x - y)$, with the initial conditions

$$f(\Lambda_0, x - y) = \nu \quad g(\Lambda_0, x - y) = 0 \quad (4.17)$$

Let us start with the computation of the effective viscosity $f(\Lambda, x - y)$, which is defined as

$$-f(\Lambda, p)p^2 = \frac{\delta^2 \Gamma_\Lambda[\hat{u}, v]}{\delta \hat{u}(\vec{p}, 0) \delta v(-\vec{p}, 0)}|_{\hat{u}=v=0}. \quad (4.18)$$

The computation of the truncated effective action (4.16) can be simplified by a rescaling of the momenta appearing into the trace, that is the loop momentum q , and the external momentum p . We rescale $f(\Lambda, q)$ too in the following way:

$$x = \frac{q}{\Lambda}; \quad y = \frac{p}{\Lambda}; \quad f(\Lambda, p) = D_0^{1/3} \Lambda^{-4/3} \phi_\Lambda(y) \quad (4.19)$$

By taking into account of the explicit and implicit dependence on the scale parameter Λ of both the $f(\Lambda, p)$ and $F_\Lambda(p)$ functions, the evolution equation for $\phi_\Lambda(y)$ reads

$$\left(\Lambda \partial_\Lambda - y \partial_y - \frac{4}{3} \right) \phi_\Lambda(y) = \frac{1}{4\pi^2} \int dx \int_{-1}^1 dt (1 - t^2) (tx^3/y - 2txy - y^2) (3 + x\partial_x) h(x) \times \\ \left((x^2 + y^2 + 2txy) \phi_\Lambda(x) \left(\phi_\Lambda \left(\sqrt{x^2 + y^2 + 2txy} \right) (x^2 + y^2 + 2txy) + \phi_\Lambda(x)x^2 \right) \right)^{-1} \quad (4.20)$$

The computation of the noise correction is similar. The $g(\Lambda, q)$ function is defined by

$$g(\Lambda, p) = -i \frac{\delta^2 \Gamma_\Lambda[\hat{u}, v]}{\delta \hat{u}(\vec{p}, 0) \delta \hat{u}(-\vec{p}, 0)}|_{\hat{u}=0}. \quad (4.21)$$

As before we rescale $g(\Lambda, q)$ in the following way

$$g(\Lambda, p) = 2D_0 \Lambda^{-3} \chi_\Lambda(y) \quad (4.22)$$

so that the evolution equation for $\chi_\Lambda(y)$ reads

$$(\Lambda \partial_\Lambda - y \partial_y - 3) \chi_\Lambda(y) = \frac{1}{4\pi^2} \int dx \int_{-1}^1 dt ((1 - t^2)y^2(x^2 + y^2 + 2t^2x^2 + 3txy) (3 + x\partial_x) h(x) \\ \left(h_\Lambda \left(\sqrt{x^2 + y^2 + 2txy} \right) + \chi_\Lambda \left(\sqrt{x^2 + y^2 + 2txy} \right) \right) \times \\ \left((x^2 + y^2 + 2txy)^2 \phi_\Lambda(x) \phi_\Lambda \left(\sqrt{x^2 + y^2 + 2txy} \right) \right. \\ \left. \left(\phi_\Lambda \left(\sqrt{x^2 + y^2 + 2txy} \right) (x^2 + y^2 + 2txy) + \phi_\Lambda(x)x^2 \right) \right)^{-1} \quad (4.23)$$

As we have seen in section 3, the Kolmogorov scale K_d is related to the viscosity and to the rate of energy dissipation by the relation $K_d \approx (\mathcal{E}/\nu^3)^{1/4}$. The viscosity ν can be expressed, then, in terms of \mathcal{E} (and so D_0) and K_d as $\nu \approx \mathcal{E}^{1/3} K_d^{-4/3}$. Remembering that the initial cutoff Λ_0 must be chosen well beyond the dissipation scale K_d , we find the initial condition for the ϕ_Λ and χ_Λ functions

$$\phi_{\Lambda_0}(y) \approx \left(\frac{\Lambda_0}{K_d} \right)^{4/3} \rightarrow \infty; \quad \chi_{\Lambda_0}(y) \approx 0. \quad (4.24)$$

The system of differential equations has been solved numerically with initial conditions (4.24). Universality in the inertial range means that the exact shape of the noise correlator should not be much important for the inertial range statistics. We have verified the validity of this claim, within our approximations, by solving the ERG flow with several different noise terms. Each noise is, of course, normalized following (3.12). In addition, we asked the noise to vanish for null momenta, in agreement with the $\Lambda \rightarrow 1/L \rightarrow 0$ prescription, and to decrease exponentially to zero for $q \gg 1/L$.

5 Results

We report here the results of our numerical solution for the ERG flow. We started at $\Lambda = \Lambda_0 = 1$ with initial conditions (4.24) and solved the differential equations with an explicit finite-difference scheme. The results are obtained after a large number of iterations ($\Lambda \ll \Lambda_0$). The numerical computations show that the solutions approach a fixed point $\phi_\Lambda(y) \rightarrow \phi(y); \chi_\Lambda(y) \rightarrow \chi(y)$. For each choice of the stirring term, this fixed point doesn't depend on the exact value of the initial condition for the viscosity term.

A possible form of the stirring term is $h(pL) = L^2 p^2 \exp(-L^2 p^2)$. In figure (1) we report the results for the fixed point functions $\phi(y)$ and $\chi(y)$ obtained with this noise.

The functions $\phi(y)$ and $\chi(y)$ are computed in the range $10^{-1} < y = \frac{p}{\Lambda} < 10^2$. Clearly the subrange $y \gg 1$ corresponds to the inertial range. As it was expected, in this region the adimensionalized effective viscosity ϕ and induced noise χ functions reach a scale-invariant regime, so that we parametrize them as

$$\phi(y)_{y \gg 1} = \sigma y^{-4/3+\alpha}; \quad \chi(y)_{y \gg 1} = \gamma y^{-3+\beta}, \quad (5.25)$$

where α and β ($\alpha \approx \beta \approx +0.37$) are anomalous exponents. The first check for the validity of our approach is the computation of the energy spectrum. The correlation function at equal long time t is computed as

$$\langle u_\alpha(\vec{p}, t) u_\beta(-\vec{p}, t) \rangle = P_{\alpha\beta}(\vec{p}) \frac{2D_0(\Lambda^{-3}h(p/\Lambda) + g(\Lambda, p))}{f(\Lambda, p)p^2} \quad (5.26)$$

By taking into account (4.19) and (4.22), in the inertial range region ($y \gg 1$) we obtain

$$E(p) = \frac{1}{2\pi^2} p^2 \frac{\chi(p/\Lambda)}{\phi(p/\Lambda)p^2} \Lambda^{-5/3} (D_0)^{2/3} = \left(\frac{1}{2\pi^2} \left(\frac{\gamma}{\sigma} \right) \left(\frac{D_0}{\mathcal{E}} \right)^{2/3} \right) \mathcal{E}^{2/3} p^{-5/3} \left(\frac{p}{\Lambda} \right)^{\beta-\alpha} \quad (5.27)$$

so that $C_K = \left(\frac{1}{2\pi^2} \left(\frac{\gamma}{\sigma} \right) \left(\frac{D_0}{\mathcal{E}} \right)^{2/3} \right)$. From data of figure (1) we can obtain the plot of the Energy spectrum by a simple ratio between the χ and ϕ data normalized by $\frac{1}{2\pi^2} \left(\frac{D_0}{\mathcal{E}} \right)^{2/3}$. In the plot (2) we have resumed the spectra obtained with several different noise shapes, while in the table T1 we list the values of C_K and of the slope $S = -5/3 + (\beta - \alpha)$ obtained with each noise.

Noise	$S (\pm 0.001)$	$C_K (\pm 0.002)$	Noise	$S (\pm 0.001)$	$C_K (\pm 0.002)$
$x^2 e^{-x^2}$	-1.666	1.124	$x^4 e^{-x^4}$	-1.666	1.146
$x^2 e^{-x^4}$	-1.667	1.267	$x^6 e^{-x^6}$	-1.666	1.660
$x^2 e^{-x^6}$	-1.666	1.417	$x^8 e^{-x^8}$	-1.666	1.767
e^{-1/x^2-x^2}	-1.667	1.489	$x^{10} e^{-x^{10}}$	-1.666	1.784
$x^4 e^{-12}$	-1.667	1.624	$x^{12} e^{-x^{12}}$	-1.666	1.785

Table T1: Values of the slope S and of the Kolmogorov constant C_K for various stirring forces

6 Comments

To sum up, in order to apply the ERG method to turbulence, we proposed an unconventional, symmetry preserving, regularization procedure. In addition, we don't need to use the noise correlator of the form $F(q) \approx q^{-y}$ which is used in quite all the previous RG approaches [10][11][12][13] (an exception is given by Brax's work [14]). Such form is clearly unphysical, as it is widely stressed [6][15].

To test the method a field truncation is applied to the ERG flow. The results obtained by this quite crude approximation are encouraging. As it is clear from the table and figure

(2), the second order inertial range statistics shows the required scale-invariance with scale exponents compatible with $-5/3$ (for the second order statistics the anomalous exponent, if any, must be very small [16]) as well as strong independence with respect to large scale energy input details. The values of the C_K constant approach the value $C_K \approx 1.78$ in the limit of very narrow noise form (for experimental results see for example [16]).

This approach is worth to be further analyzed, for instance, with higher orders field truncations or, more likely, with nonpolynomial approximations schemes.

In a next paper [17] the complete analysis of such procedure will be reported and different approximation schemes will be considered.

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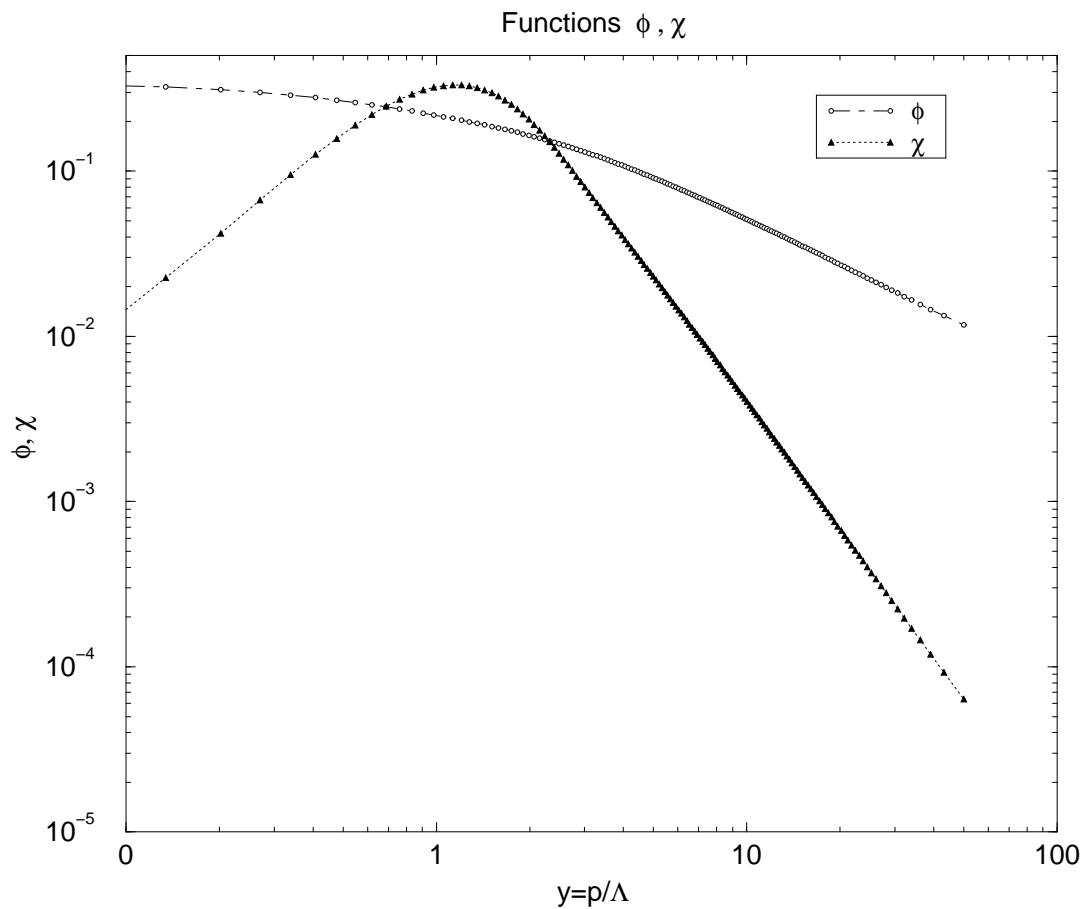


Figure 1: ϕ and χ functions

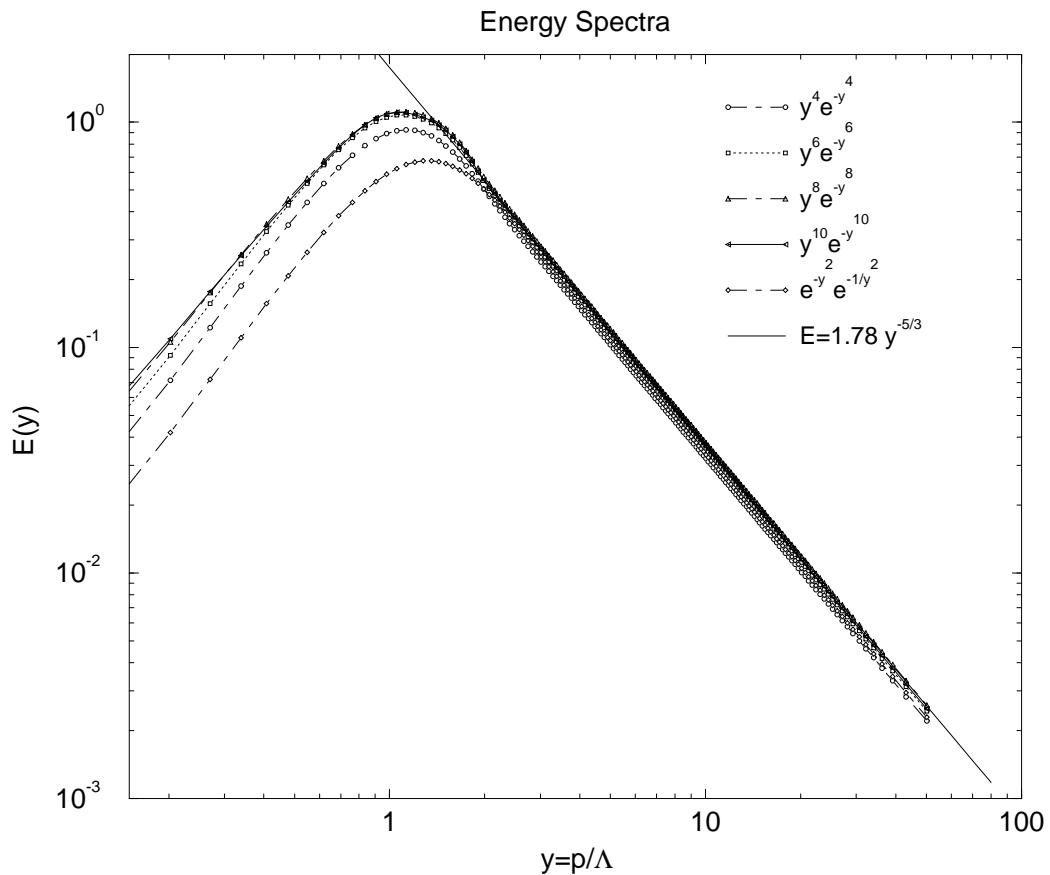


Figure 2: Energy spectra for various noise terms